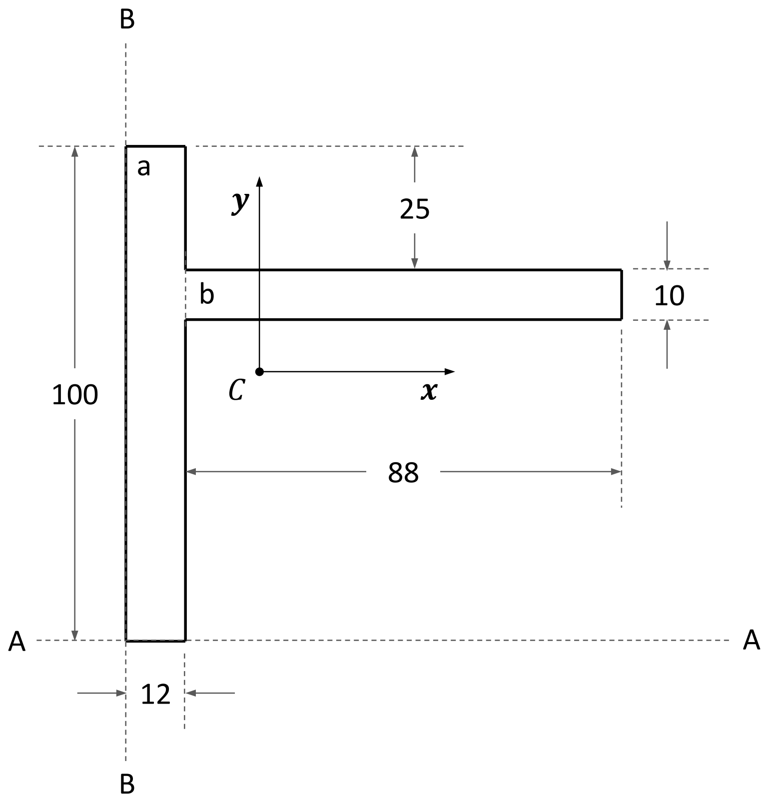
2013-2014 MM2MS3 Exam Solutions

1.

(a)



Total area,

$$A = (12 \times 100)_a + (88 \times 10)_b = 2080 \text{ mm}^2$$

[2 marks]

Taking moments about AA:

$$\bar{y} = \frac{(12 \times 100 \times 50)_a + (88 \times 10 \times 70)_b}{2080} = \mathbf{58.46\ mm}$$

[2 marks]

Similarly, taking moments about BB:

$$\bar{x} = \frac{(100 \times 12 \times 6)_a + (10 \times 88 \times 56)_b}{2080} = \mathbf{27.15\ mm}$$

[2 marks]

(b)

Therefore, using the Parallel Axis Theorem,

$$I_{x'} = (I_x + Ab^2)_a + (I_x + Ab^2)_b$$

$$= \left(\frac{12 \times 100^3}{12} + 12 \times 100 \times (50 - 58.46)^2\right) + \left(\frac{88 \times 10^3}{12} + 88 \times 10 \times (70 - 58.46)^2\right)$$

$$= 1{,}210{,}410.26 \text{ mm}^4$$

[2 marks]

and,

$$I_{y'} = \left(I_y + Aa^2\right)_a + \left(I_y + Aa^2\right)_b$$

$$= \left(\frac{100 \times 12^3}{12} + 100 \times 12 \times (6 - 27.15)^2\right) + \left(\frac{10 \times 88^3}{12} + 10 \times 88 \times (56 - 27.15)^2\right)$$

$$= 1{,}851{,}524.13 \text{ mm}^4$$

[2 marks]

Also,

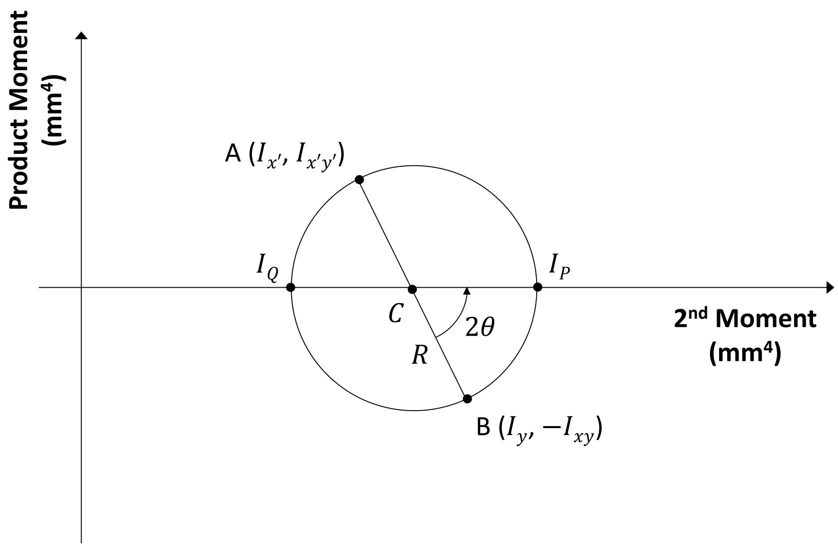
$$I_{x'y'} = \left(I_{xy} + Aab\right)_a + \left(I_{xy} + Aab\right)_b$$

$$= \big(0 + 12 \times 100 \times (6 - 27.15) \times (50 - 58.46)\big) + \big(0 + 88 \times 10 \times (56 - 27.15) \times (70 - 58.46)\big)$$

$$= 507{,}692.32 \text{ mm}^4$$

[2 marks]

Mohr's Circle,



[3 marks]

$$Centre, C = \frac{I_{x\prime} + I_{y\prime}}{2} = \frac{1{,}210{,}410.26 + 1{,}851{,}524.13}{2} = 1{,}530{,}967.2\ \mathrm{mm}^4$$

$$Radius, R = \sqrt{\left(\frac{I_{x'} - I_{y'}}{2}\right)^2 + I_{x'y'}{}^2} = \sqrt{\left(\frac{1{,}210{,}410.26 - 1{,}851{,}524.13}{2}\right)^2 + 507{,}692.32^2} = 600{,}423.38\ \mathrm{mm}^4$$

[2 marks]

Therefore, the Principal 2nd Moments of Area are:

$$I_P = C + R = 1{,}530{,}967.2 + 600{,}423.38 = \mathbf{2,131,390.58\ mm^4}$$

and,

$$I_Q = C - R = 1{,}530{,}967.2 - 600{,}423.38 = \mathbf{930,543.82\ mm^4}$$

[2 marks]

(c)

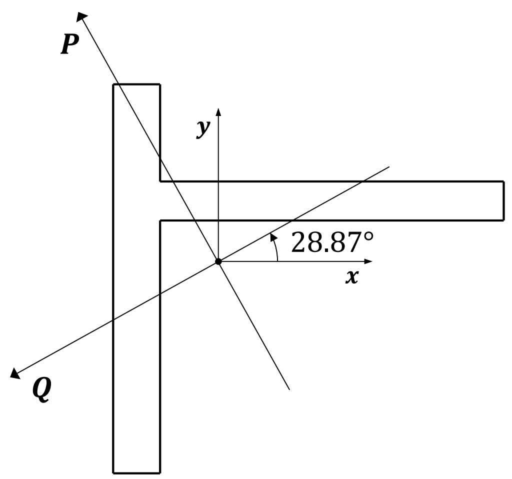
From the Mohr's circle above:

$$sin2\theta = \frac{-I_{x'y'}}{R} = \frac{507{,}692.32}{600{,}423.38}$$

$$\therefore \boldsymbol{\theta = -0.504 rad = -28.87°}$$

[3 marks]

Therefore the Principal Axes are at -28.87° (anti-clockwise) from the x-y axes, as shown on the diagram below.



[3 marks]

2.

(a)

Figure Q2.1 shows an element of a straight beam, length δs, which bends to curvature R, due to an applied bending moment M. The angle subtended by the element of beam is $\delta\phi$, also equal to the change in slope of the beam over δs.

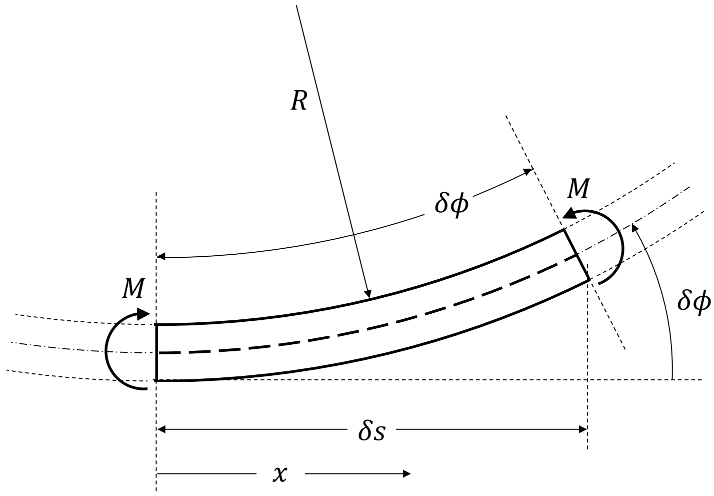


Fig Q2.1 Element of a Beam

[1 mark]

Therefore, the strain energy (work done) for the element, δU, is given by (area under the curve in Fig Q2.2):

$$\delta U = \frac{1}{2} M\delta\phi \qquad (1)$$

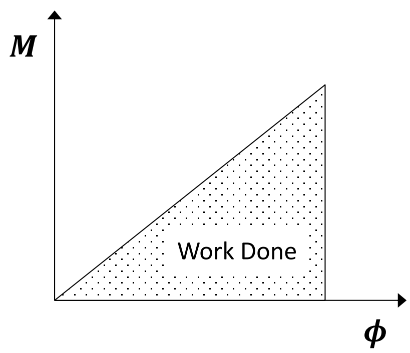


Fig Q2.2 Plot of Strain Energy in a Beam

[1 mark]

Equation of an arc:

$$\delta s = R\delta\phi \qquad (2)$$

and Beam Bending equation:

$$\frac{M}{I} = \frac{E}{R} \qquad (3)$$

Therefore, rearranging (3) for R and substituting this into (2):

$$\delta s = \frac{EI}{M}\delta\phi$$

$$\therefore\ \delta\phi = \frac{M}{EI}\delta s$$

Substituting this into (1) gives:

$$\delta U = \frac{M^2}{2EI}\delta s \qquad (4)$$

[1 mark]

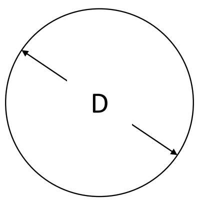
Thus, for a beam of length, L, integrating (4) across this length gives:

$$\boldsymbol{U = \int_0^L \frac{M^2}{2EI}\delta x}$$

[2 marks]

(b)

Second Moment of Area, I, calculation:

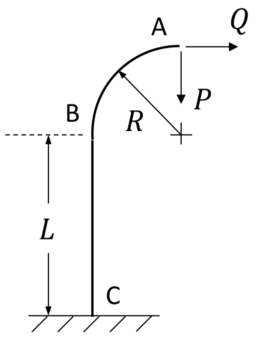


Beam cross-section

$$I = \frac{\pi D^4}{64} = \frac{\pi \times 40^4}{64} = 125{,}663.71\ \mathrm{mm}^4$$

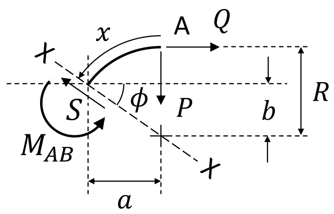
[1 mark]

Adding dummy load, Q, and labelling the structure:



[1 mark]

Free body diagram for section AB (*bending only*):



[2 marks]

Taking moments about X-X:

$$M_{AB} = Pa + Q(R - b) = PRcos\phi + Q(R - Rsin\phi) = R\big(Pcos\phi + Q(1 - sin\phi)\big)$$

[1 mark]

Substituting this into the equation for Strain Energy in a beam under bending gives,

$$U_{AB} = \int \frac{{M_{AB}}^2}{2EI} ds = \int_0^{\pi/2} \frac{\Big(R\big(Pcos\phi + Q(1 - sin\phi)\big)\Big)^2}{2EI} Rd\phi$$

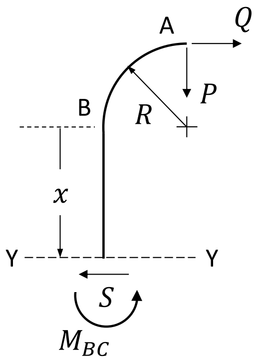
where,

$$dx = Rd\phi$$

$$\therefore U_{AB} = \frac{R^3}{2EI} \int_0^{\pi/2} \big(Pcos\phi + Q(1 - sin\phi)\big)^2 d\phi$$

[1 mark]

Free body diagram for section BC (*bending only*):



[2 marks]

Taking moments about Y-Y:

$$M_{BC} = PR + Q(R + x)$$

[1 mark]

Substituting this into the equation for Strain Energy in a beam under bending gives,

$$U_{BC} = \int \frac{{M_{BC}}^2}{2EI} ds = \frac{1}{2EI} \int_0^L \big(PR + Q(R + x)\big)^2 dx$$

[1 mark]

Total Strain Energy:

$$U = U_{AB} + U_{BC} = \frac{R^3}{2EI} \int_0^{\pi/2} \big(Pcos\phi + Q(1 - sin\phi)\big)^2 d\phi + \frac{1}{2EI} \int_0^L \big(PR + Q(R + x)\big)^2 dx \qquad (5)$$

[2 marks]

Vertical deflection at position A, u_{v_A}

Differentiating (5) with respect to applied load, P:

$$u_{v_A} = \frac{\partial U}{\partial P} = \frac{R^3}{EI} \int_0^{\pi/2} \big(Pcos\phi + Q(1 - sin\phi)\big)(cos\phi) d\phi + \frac{1}{EI} \int_0^L \big(PR + Q(R + x)\big)(R) dx$$

[1 mark]

Setting Dummy Load, Q, to 0:

$$u_{v_A} = \frac{PR^3}{EI}\int_0^{\pi/2} cos^2\phi d\phi + \frac{PR^2}{EI}\int_0^L 1dx \tag{6}$$

Trigonometric Identities:

$$sin^2\phi + cos^2\phi = 1 \tag{7}$$

and,

$$cos2\phi = cos^2\phi - sin^2\phi \tag{8}$$

Rearranging (7) gives,

$$sin^2\phi = 1 - cos^2\phi$$

[1 mark]

Substituting this into (8) gives,

$$cos2\phi = cos^2\phi - (1 - cos^2\phi) = 2cos^2\phi - 1$$

$$\therefore cos^2\phi = \frac{1}{2}(cos2\phi + 1)$$

Substituting this into (6) gives,

$$u_{v_A} = \frac{PR^3}{EI}\int_0^{\pi/2} \frac{1}{2}(cos2\phi + 1)d\phi + \frac{PR^2}{EI}\int_0^L 1dx = \frac{PR^3}{2EI}\left[\frac{sin2\phi}{2} + \phi\right]_0^{\pi/2} + \frac{PR^2}{EI}[x]_0^L$$

$$= \frac{PR^3}{2EI}\left(\frac{sin(\pi)}{2} + \frac{\pi}{2}\right) + \frac{PR^2}{EI}(L) = \frac{\pi PR^3}{4EI} + \frac{PR^2L}{EI}$$

$$\therefore u_{v_A} = \frac{PR^2}{EI}\left(\frac{\pi R}{4} + L\right)$$

Substituting values of P, R, E, I and L into this gives:

$$\boldsymbol{u_{v_A} = 1.055\ \mathrm{mm}}$$

(downwards deflection, i.e. in the direction of the load, P)

[2 marks]

Horizontal deflection at position A, u_{h_A}

Differentiating (5) with respect to dummy load, Q:

$$u_{h_A} = \frac{\partial U}{\partial Q} = \frac{R^3}{EI}\int_0^{\pi/2} \big(Pcos\phi + Q(1 - sin\phi)\big)(1 - sin\phi)d\phi + \frac{1}{EI}\int_0^{L} \big(PR + Q(R + x)\big)(R + x)dx$$

[1 mark]

Setting Dummy Load, Q, to 0:

$$u_{h_A} = \frac{\partial U}{\partial Q} = \frac{PR^3}{EI}\int_0^{\pi/2} (cos\phi - cos\phi sin\phi)d\phi + \frac{PR}{EI}\int_0^{L} (R + x)dx \qquad (9)$$

Trigonometric Identity:

$$sin2\phi = 2sin\phi cos\phi$$

$$\therefore sin\phi cos\phi = \frac{1}{2}sin2\phi$$

[1 mark]

Substituting this into (9) gives,

$$u_{h_A} = \frac{PR^3}{EI}\int_0^{\pi/2} \left(cos\phi - \frac{1}{2}sin2\phi\right)d\phi + \frac{PR}{EI}\int_0^{L} (R + x)dx$$

$$= \frac{PR^3}{EI}\left[sin\phi + \frac{1}{4}cos2\phi\right]_0^{\pi/2} + \frac{PR}{EI}\left[Rx + \frac{x^2}{2}\right]_0^{L}$$

$$= \frac{PR^3}{EI}\left(\left(sin\left(\frac{\pi}{2}\right) + \frac{1}{4}cos(\pi)\right) - \left(\frac{1}{4}cos(0)\right)\right) + \frac{PR}{EI}\left(RL + \frac{L^2}{2}\right) = \frac{PR^3}{2EI} + \frac{PRL}{EI}\left(R + \frac{L}{2}\right)$$

$$\therefore u_{h_A} = \frac{PR}{EI}\left(\frac{R^2}{2} + L\left(R + \frac{L}{2}\right)\right)$$

Substituting values of P, R, E, I and L into this gives:

$$\boldsymbol{u_{h_A} = 1.71\ \mathrm{mm}}$$

(deflection to the right, i.e. in the direction of the load, Q)

[2 marks]

3.

(a)

Inner radius of flywheel increases by u_1
outer radius of flywheel decreases by u_2.

$u_1 + u_2 = 0.5$ mm (A)

On steel shaft surface $\sigma_\theta = \sigma_r = -p$.

At flywheel bore ($r = 25$) $\sigma_r = -p$

$$-p = A - \frac{B}{25^2} \quad (1)$$

At flywheel outer ($r = 100$) $\sigma_r = 0$

$$0 = A - \frac{B}{100^2} \quad (2)$$

(2) − (1) gives $B = \frac{2000}{3} p$.

sub in (2) $A = \frac{2}{30} p$

For flywheel

$$\sigma_r = \frac{2}{30} p - \frac{1}{r^2}\left(\frac{2000}{3} p\right)$$

$$\sigma_r = \frac{2}{30} p \left(1 - \frac{10{,}000}{r^2}\right)$$

$$\sigma_\theta = \frac{2}{30} p \left(1 + \frac{10{,}000}{r^2}\right)$$

For steel shaft

$$\varepsilon_\theta = \frac{u}{r} = \frac{1}{E_{st}}\left(\sigma_\theta - \nu_{st}(\sigma_r + \sigma_z)\right)$$

$$\varepsilon_\theta = \frac{-u_2}{25} = \frac{1}{200\times10^3}\left(-p - \nu_{st}(-p+0)\right)$$

$$= \frac{-p\,(0.7)}{200\times10^3}$$

For flywheel.

$$\varepsilon_\theta = \frac{u_1}{25} = \frac{1}{70\times10^3}\left(\frac{34}{30}p - 0.33(-p)\right)$$

$$= \frac{44\,p}{210\times10^3}$$

Using Ⓐ

$$25\left(\frac{44\,p}{210\times10^3} + \frac{0.7\,p}{200\times10^3}\right) = 0.5$$

$$\underline{p = 93.9\ \text{MPa}}$$

(b)

In flywheel

<u>At r = 25</u>

$$\sigma_r = -p = -94\ \text{MPa}$$

$$\sigma_\theta = A + \frac{B}{r^2} = p\left(\frac{2}{30} + \frac{2000}{3}\left(\frac{1}{25^2}\right)\right)$$

$$= 106.5\ \text{MPa}$$

<u>at r = 62.5 (mid point)</u>

$$\sigma_r = A - \frac{B}{r^2} = -9.8\ \text{MPa}$$

$$\sigma_\theta = A + \frac{B}{r^2} = 22.3\ \text{MPa}$$

at r = 100

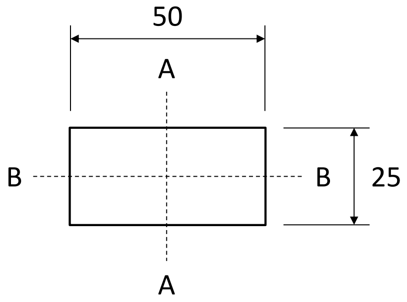
$$\sigma_r = 0$$

$$\sigma_\theta = 12.5\ \text{MPa}$$

Present as sketch σ vs r.

4.

(a)

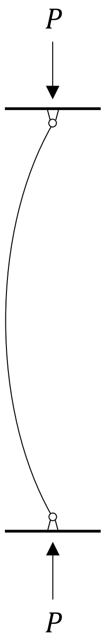


All dimensions in mm

$$I_{min} = I_{BB} = \frac{bd^3}{12} = \frac{50 \times 25^3}{12} = 65{,}104.17 \text{ mm}^4$$

[2 marks]

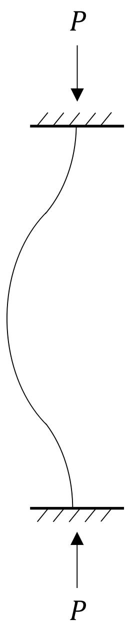
(i) Hinged-hinged



$$P_{crit} = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 \times 209{,}000 \times 65{,}104.17}{1000^2} = 134{,}293.45 \text{ N} = \mathbf{134.29\ kN}$$

[2 marks]

(ii) Fixed-fixed



$$P_{crit} = \frac{4\pi^2 EI}{l^2} = \frac{4 \times \pi^2 \times 209{,}000 \times 65{,}104.17}{1000^2} = 537{,}173.81 \text{ N}$$

$$= \mathbf{537.17\ kN}$$

[2 marks]

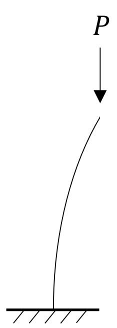
(iii) Free-fixed

$$P_{crit} = \frac{2.045\pi^2 EI}{l^2} = \frac{\pi^2 \times 209{,}000 \times 65{,}104.17}{4 \times 1000^2} = 33{,}573.36 \text{ N}$$

$$= \mathbf{33.57\ kN}$$

[2 marks]

(iv) Fixed-hinged



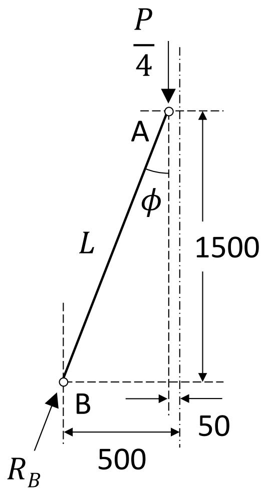
$$P_{crit} = \frac{2.045\pi^2 EI}{l^2} = \frac{2.045 \times \pi^2 \times 209{,}000 \times 65{,}104.17}{1000^2} = 274{,}630.11 \text{ N}$$

$$= \mathbf{274.63\ kN}$$

[2 marks]

(b)

Considering one of the tripod legs:



[3 marks]

$$L = \sqrt{1500^2 + (500 - 50)^2} = 1{,}566.05 \text{ mm}$$

[2 marks]

Buckling:

$$P_{crit} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 \times E \times \frac{\pi D^4}{64}}{L^2}$$

[3 marks]

Plastic collapse:

$$P_p = \sigma_y A = \frac{\sigma_y \pi D^2}{4}$$

[3 marks]

For the load, P, to be equal to give failure by buckling and by plastic collapse:

$$P = P_{crit} = P_p$$

$$\therefore \frac{\pi^2 \times E \times \frac{\pi D^4}{64}}{L^2} = \frac{\sigma_y \pi D^2}{4}$$

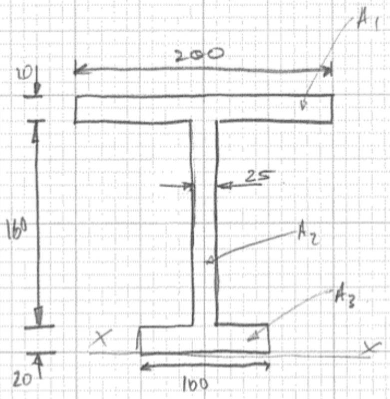
$$\therefore D^2 = \frac{16 \sigma_y L^2}{\pi^2 E}$$

$$\therefore D = \frac{4L}{\pi}\sqrt{\frac{\sigma_y}{E}} = \frac{4 \times 1{,}566.05}{\pi}\sqrt{\frac{250}{209{,}000}}$$

$$\therefore \boldsymbol{D = 68.96\ \mathrm{mm}}$$

[4 marks]

5.



$S = 50kN.$

N.A. Position:

$A_1 = 200 \times 20 = 4000 mm^2$

$A_2 = 160 \times 25 = 4000 mm^2$

$A_3 = 100 \times 20 = 2000 mm^2$

$A_T = A_1 + A_2 + A_3 = 10000 mm^2$

Area moment about x-x $\quad Q_1 = 190 \times 4000 = 760\,000 mm^3$

$Q_2 = 100 \times 4000 = 400\,000 mm^3$

$Q_3 = 10 \times 2000 = 20\,000 mm^3$

$Q_T = Q_1 + Q_2 + Q_3 = 1\,180\,000 mm^3$

centroid pos from x-x $= \dfrac{Q_T}{A_T} = \underline{\underline{118 mm}}$

a) Moments of inertia about N.A

$$I_1 = \frac{bd^3}{12} + A\bar{y}^2 = \frac{200 \times 20^3}{12} + 4000 \times (190 - 118)^2$$

$$= \underline{2.08 \times 10^7 mm^4}$$

$$I_2 = \frac{25 \times 160^3}{12} + 4000 \times (100 - 118)^2$$

$$= \underline{9.83 \times 10^7 mm^4}$$

$$I_3 = \frac{100 \times 20^3}{12} + 2000 \times (10 - 118)^2$$

$$= \underline{2.34 \times 10^7 mm^4}$$

$$I_T = I_1 + I_2 + I_3 = \underline{\underline{1.425 \times 10^8 mm^4}}$$

[7 marks]

b) Vertical Shear

@ A & E, both free surfaces so $\tau = 0$.

@ B, step change in τ due to change in section ∴ 2 values -

$$\tau = \frac{SQ}{Iz}$$

where S = Shear Force
$Q = A\bar{y}$ (of area above (or below) point)
I = moment of area of entire x-section
z = section thickness at location/point

Flange shear stress at B

$$\frac{50000 \times 200 \times 20 \times (190-118)}{1.425 \times 10^{6} \times 200} = \underline{\underline{0.51\,MPa}}$$

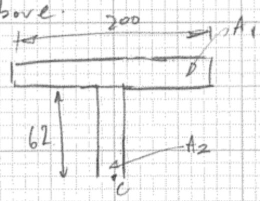
Web stress at B.

$$\frac{50\,000 \times 200 \times 20 \times (190-118)}{1.425 \times 10^{8} \times 25} = \underline{\underline{4.04\,MPa}}$$

Web stress at C

Need to consider 2 areas above.

∴ $Q = \sum A_x \bar{y}_x$



$= 200 \times 20 \times (190-118)$
$+ 25 \times 62 \times 31$
$= \underline{3.36 \times 10^5}$

$$\therefore \frac{50000 \times 3.36 \times 10^5}{1.425 \times 10^8 \times 25} = \underline{\underline{4.72\,MPa}}$$

Web stress at D (use area below for simplicity)

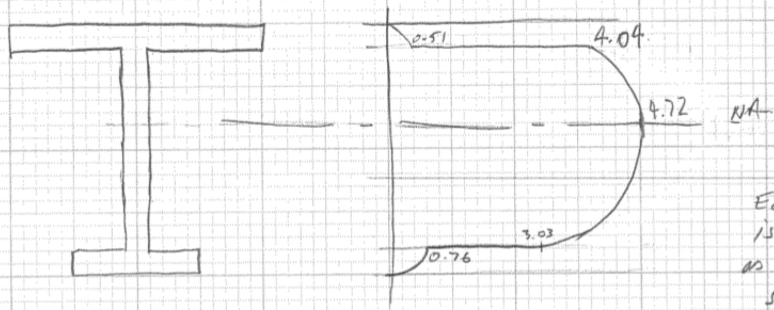
$$\frac{50\,000 \times 100 \times 20 \times (118-10)}{1.425 \times 10^8 \times 25} = \underline{\underline{3.03\,MPa}}$$

Flange stress at D

$$\frac{50000 \times 100 \times 20 \times (118-10)}{1.425 \times 10^8 \times 100} = \underline{\underline{0.76\,MPa}}$$

[13 marks]

c) Sketch.



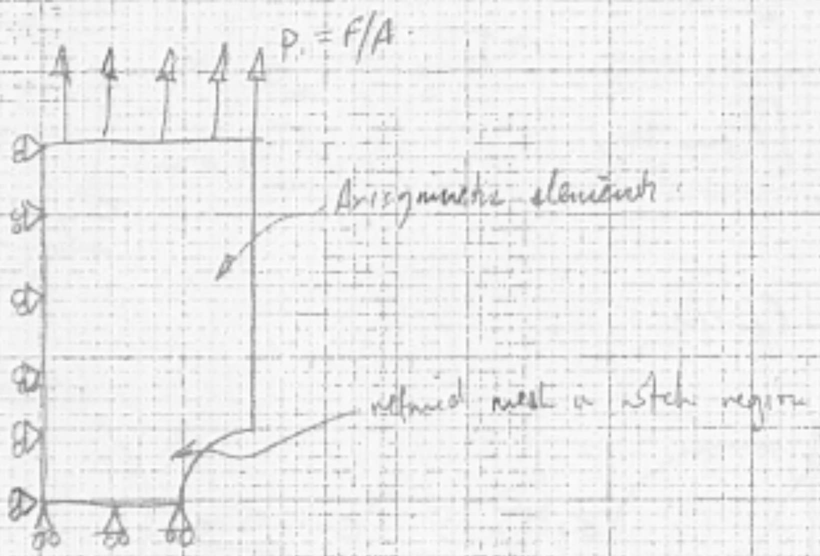
Each section is parabolic as rectangular sections

[5 marks]

6.

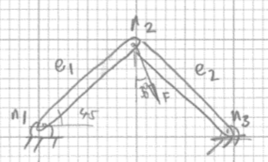
a) * Axisymmetric elements.

* Symmetry boundary conditions



[7 Marks]

b) i)



$$[k]_{e_1} = \frac{AE}{L} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}$$

$$[k]_{e_2} = \frac{AE}{L} \begin{array}{c} \begin{array}{cccc} u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} u_3 \\ u_4 \\ u_5 \\ u_6 \end{array}$$

$$[k]_g = \frac{AE}{L} \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 & 0 & 0 \\ 0.5 & 0.5 & -0.5 & -0.5 & 0 & 0 \\ -0.5 & -0.5 & 1 & 0 & -0.5 & 0.5 \\ -0.5 & -0.5 & 0 & 1 & 0.5 & -0.5 \\ 0 & 0 & -0.5 & 0.5 & 0.5 & -0.5 \\ 0 & 0 & 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

[10 Marks]

b) ii

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ F\sin 30 \\ -F\cos 30 \\ F_5 \\ F_6 \end{Bmatrix} \qquad \{F\} = [k]\{u\} \qquad \{u\} = \begin{Bmatrix} 0 \\ 0 \\ u_3 \\ u_4 \\ 0 \\ 0 \end{Bmatrix}$$

gives:

$$\begin{Bmatrix} F\sin 30 \\ -F\cos 30 \end{Bmatrix} = \begin{bmatrix} \frac{AE}{L} & 0 \\ 0 & \frac{AE}{L} \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

expressions disps:

$$u_3 = F\sin 30 \times L/AE = \underline{\underline{0.5FL/AE}}$$

$$u_4 = \underline{\underline{-F\cos 30 \times L/AE}} = \underline{\underline{-\sqrt{3}/2\, FL/AE}}$$

[3 Marks]

b) iii)

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_5 \\ F_6 \end{Bmatrix} = \begin{bmatrix} -\frac{0.5AE}{L} & -\frac{0.5AE}{L} \\ -\frac{0.5AE}{L} & -\frac{0.5AE}{L} \\ -\frac{0.5AE}{L} & \frac{0.5AE}{L} \\ \frac{0.5AE}{L} & -\frac{0.5AE}{L} \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

$$\Rightarrow \quad F_1 = -\frac{0.5AE}{L}u_3 - \frac{0.5AE}{L}u_4 \quad \Rightarrow \quad F_1 = -0.25F + \frac{\sqrt{3}}{4}F = \underline{\underline{0.183F}}$$

$$F_2 = \text{''} \quad \text{''} \quad \Rightarrow \quad \text{''} \quad \text{''} = \underline{\underline{0.183F}}$$

$$F_5 = -\frac{0.5AE}{L}u_3 + \frac{0.5AE}{L}u_4 \quad \Rightarrow \quad -0.25F - \frac{\sqrt{3}}{4}F = \underline{\underline{-0.68F}}$$

$$F_6 = \frac{0.5AE}{L}u_3 - \frac{0.5AE}{L}u_4 \quad \Rightarrow \quad 0.25F + \frac{\sqrt{3}}{4}F = \underline{\underline{0.68F}}$$

[5 Marks]